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STUDIES OF BOUNDARY LAYER SLIP SOLUTIONS AND ALDEN'S METHOD FOR BOUNDARY LAYER CORRECTION

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by

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FLUID FLOW AND
HEAT TRANSFER
AT LOW PRESSURES
AND TEMPERATURE

STUDIES OF BOUNDARY LAYER SLIP SOLUTIONS AND ALDEN'S METHOD FOR BOUNDARY LAYER CORRECTION

by

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ABSTRACT

Two aspects of incompressible laminar boundary layer flow on a semi-infinite flat plate at zero incidence are discussed. Firstly, Alden's proposed scheme of solution of the Navier-Stokes equations for this problem in terms of an expansion in inverse powers of the local Reynolds number is demonstrated to be invalid, inasmuch as the boundary conditions in the free stream cannot be satisfied to all orders. Secondly, an account is given of the effect of a velocity slip boundary condition on the solution of the Oseen boundary layer equations for the same problem. It is found that an additional non-uniformity in the solution is caused by the introduction of slip.

NOMENCLATURE

C_D	Drag coefficient, eq. (44)
C_f	Local skin friction coefficient, eq. (42)
C_{11}, C_{22}	Constants defined in eq. (18)
C_{ni}	Constants defined in eqs. (23) and (30)
$F_n(\xi)$	Functions defined in eq. (14)
$F_{ni}(\xi)$	Functions defined in eqs. (24), (25), (28) and (30)
$f_n(\xi)$	Functions defined in eq. (12)
$g_n(\xi)$	Functions defined in eq. (58)
$h_n(\xi)$	Functions defined in eq. (26)
K	Constant defined in eq. (32) $(K = K_{23})$
K_n	Knudsen number
K_{nn}	Constants of integration (Appendix II)
$O(x^{-N})$	Symbolizes terms of order $-N$, i.e., $\lim_{x\to\infty} x^N O(x^{-N})$ = constant
p	Pressure
r	$= \sqrt{x^2+y^2}$
Re_L	$= \frac{u_\infty L}{\nu_\infty}$, Reynolds number based on L
Re_x	$= \frac{u_\infty x}{\nu_\infty}$, Reynolds number based on x
s	$= u_\infty/\sqrt{2RT_\infty}$, molecular speed ratio
u, v	x and y velocity components
$u_n(x,y)$	Functions defined in eq. (46)
u_∞	Free stream velocity
x, y	Cartesian coordinates
α_n	Constants defined in eq. (29)
β	Constant defined in eq. (21)
Γ	Gamma function

NOMENCLATURE (continued)

ζ		Slip parameter, see eq. (36)
η	=	$\tau\nu^{-\frac{1}{2}}$, auxiliary variable
θ	=	$\tan^{-1} y/x$
μ		Viscosity
ν	=	μ/ρ , kinematic viscosity
ν_∞		Free stream kinematic viscosity
ξ	=	$\eta u_\infty^{-\frac{1}{2}}$, auxiliary variable
ρ		Density
ρ_∞		Free stream density
σ, τ		Parabolic coordinates, eq. (6)
ψ		Stream function, eq. (2)
$\psi_n(\sigma,\eta)$		Functions defined in eq. (11)
∇^2		Laplacian operator
$\backsim$		Asymptotic equality

1.0 INTRODUCTION

Blasius solved the incompressible boundary layer problem for laminar flow past a semi-infinite flat plate at zero angle of attack (references 1 and 2) and his result describes the flow quite well for regions where the local Reynolds number, $u_\infty x / \nu$, is large. Von Karman has proposed (reference 3) that the Blasius solution be considered the first term of an asymptotic representation of a solution to the full Navier-Stokes equations; calculation of the second term should extend boundary layer theory to lower Reynolds numbers. Due to the non-uniformities which are inherent in boundary layer theory, any such representation will contain terms which express these non-uniformities; in particular, one would not expect a simple power series in the viscosity. The specific form for the stream function expansion suggested by von Karman has a line singularity on the y-axis (the plate is assumed to occupy the positive x-axis with the leading edge at the origin). Alden (reference 4) improved this situation somewhat by modifying von Karman's representation in such a way as to rotate the line singularity to the negative x-axis. Adopting von Karman's philosophy, he obtained a "first correction" to the Blasius solution. At first glance, Alden's iteration scheme seems to lead to an asymptotic series which is formally a solution of the Navier-Stokes problem. In the first part of this paper it will be shown that Alden's method cannot lead to such a solution because of an unavoidable violation of the free stream boundary conditions.

In the case of a gas at low Reynolds numbers the boundary conditions at a solid surface must be modified to account for the slipping of the gas at the wall (references 5 through 9). To support the conjecture that slip introduces an additional non-uniformity into the solution, the linearized boundary layer problem (for the flat plate) with a slip condition is examined in detail in the second part of this paper. An explicit solution is known and this is compared with the divergent series solution obtained by a perturbation in the "slip parameter".

2.0 ANALYSIS

2.1 Alden's Method

The Navier-Stokes equations of motion and continuity, specialized for the steady state two dimensional flow of an incompressible fluid are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \tag{1}$$

where x and y are cartesian coordinates, u and v are the cartesian velocity components, ∇^2 is the Laplacian operator, and ρ , p and ν are, respectively, the density, the pressure and the kinematic viscosity.

A stream function, ψ , may be introduced so that

$$u = \frac{\partial \psi}{\partial y} \; ; \quad v = -\frac{\partial \psi}{\partial x} \tag{2}$$

and the continuity equation is automatically satisfied. If the momentum equations are cross differentiated and subtracted, to eliminate the pressure terms, one obtains, in terms of the stream function

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \nu\nabla^2\nabla^2\psi \tag{3}$$

Physical arguments lead one to suspect that the boundary layer equation (reference 10)

$$\frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x\,\partial y^2} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial y^3} = \nu\frac{\partial^4 \psi}{\partial y^4} \tag{4}$$

may describe the flow quite well in regions wher the local Reynolds number, $u_\infty x/\nu$, is large, (u_∞ being the free stream velocity).

To complete this stream function formulation of the incompressible boundary layer problem for laminar flow past a semi-infinite flat plate at zero angle of attack, it will be assumed that the plate occupies the positive x-axis with the leading edge at the origin and it will be required that

$$\left.\begin{aligned} &\psi = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0 \quad \text{at } y = 0,\ x > 0 \\ &\left.\begin{aligned} &\frac{\partial\psi}{\partial x} \to 0 \\ &\frac{\partial\psi}{\partial y} \to u_\infty \end{aligned}\right\} \text{ as } r \to \infty,\ 0 < \theta < \pi \end{aligned}\right\} \tag{5}$$

where r and θ are polar coordinates with origin at the leading edge of the plate. This is essentially the problem solved by Blasius.

In terms of parabolic coordinates,

$$x + iy = (\sigma + i\tau)^2 = re^{i\theta} \tag{6}$$

equation (3) becomes

$$\begin{aligned} &(\sigma^2+\tau^2)\left\{\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\sigma}\nabla^2\psi - \frac{\partial\psi}{\partial\sigma}\frac{\partial}{\partial\tau}\nabla^2\psi\right\} \\ &-2\nabla^2\psi\left\{\sigma\frac{\partial\psi}{\partial\tau} - \tau\frac{\partial\psi}{\partial\sigma}\right\} \\ &= \nu\left\{4 - 4\left(\sigma\frac{\partial}{\partial\sigma} + \tau\frac{\partial}{\partial\tau}\right) + (\sigma^2+\tau^2)\nabla^2\right\}\nabla^2\psi \end{aligned} \tag{7}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial\sigma^2} + \dfrac{\partial^2}{\partial\tau^2}$

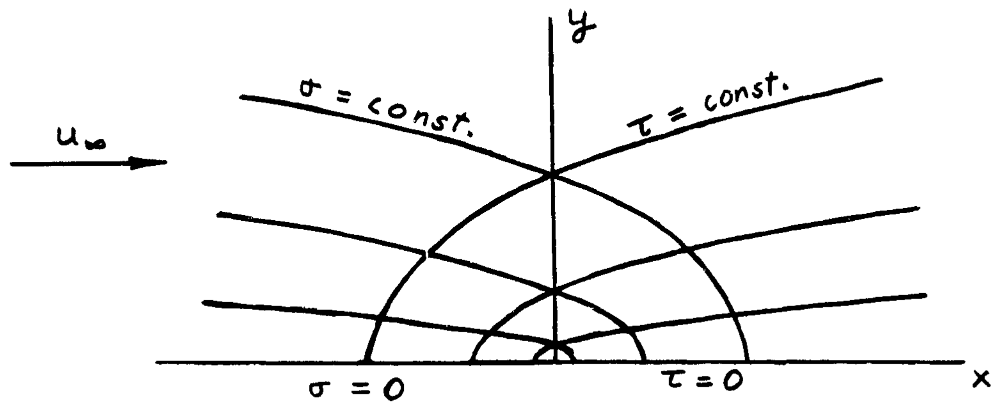

Figure 1.

Applying the analogous boundary layer arguments to equation (7) one obtains the "parabolic boundary layer equation",

$$\sigma^2 \frac{\partial\psi}{\partial\tau}\frac{\partial^3\psi}{\partial\sigma\partial\tau^2} - \sigma^2\frac{\partial\psi}{\partial\sigma}\frac{\partial^3\psi}{\partial\tau^3} - 2\sigma\frac{\partial^2\psi}{\partial\tau^2}\frac{\partial\psi}{\partial\tau} = \nu\sigma^2\frac{\partial^4\psi}{\partial\tau^4} \tag{8}$$

Note that the parabolic boundary layer equation is not the same as the cartesian boundary layer equation expressed in parabolic coordinates (see reference 11).

H. L. Alden (reference 4), following a suggestion by von Karman (reference 3), attempted to improve the boundary layer solution by developing the iteration scheme which solves equation (1) with the Blasius solution as the first term in an expansion. Using his scheme, Alden found two terms in a proposed expansion. The first term is the solution of the parabolic boundary layer problem and the second term may be thought of as a correction term. Since there is no obvious mathematical reason for concluding that his scheme will not lead to an asymptotic expansion which is formally a solution of the problem, it is worth while to investigate further. Alden introduces the auxiliary variables

$$\left.\begin{aligned} \eta &= \tau\nu^{-\frac{1}{2}} \\ \xi &= \eta u_\infty^{-\frac{1}{2}} \end{aligned}\right\} \tag{9}$$

and rewrites equation (7) as

$$
\begin{aligned}
&(\sigma^2+\nu\eta^2)\left\{\frac{\partial\psi}{\partial\eta}\left(\frac{\partial^3\psi}{\partial\sigma^3}+\nu^{-1}\frac{\partial^3\psi}{\partial\sigma\partial\eta^2}\right)-\frac{\partial\psi}{\partial\sigma}\left(\frac{\partial^3\psi}{\partial\sigma^2\partial\eta}+\nu^{-1}\frac{\partial^3\psi}{\partial\eta^3}\right)\right\}\\
&-2\left(\frac{\partial^2\psi}{\partial\sigma^2}+\nu^{-1}\frac{\partial^2\psi}{\partial\eta^2}\right)\left(\sigma\frac{\partial\psi}{\partial\eta}-\nu\eta\frac{\partial\psi}{\partial\sigma}\right)\\
&-\nu^{3/2}\left\{4\left(\frac{\partial^2\psi}{\partial\sigma^2}+\nu^{-1}\frac{\partial^2\psi}{\partial\eta^2}\right)-4\sigma\left(\frac{\partial^3\psi}{\partial\sigma^3}+\nu^{-1}\frac{\partial^3\psi}{\partial\sigma\partial\eta^2}\right)\right.\\
&\qquad-4\eta\left(\frac{\partial^3\psi}{\partial\sigma^2\partial\eta}+\nu^{-1}\frac{\partial^3\psi}{\partial\eta^3}\right)\\
&\qquad\left.+(\sigma^2+\nu\eta^2)\left(\frac{\partial^4\psi}{\partial\sigma^4}+2\nu^{-1}\frac{\partial^4\psi}{\partial\sigma^2\partial\eta^2}+\nu^{-2}\frac{\partial^4\psi}{\partial\eta^4}\right)\right\}\\
&=0
\end{aligned}
\tag{10}
$$

Alden proposed the iteration scheme whereby $\psi(\sigma,\eta,\nu)$ is formally expressed as

$$
\psi(\sigma,\eta,\nu)=\nu^{1/2}\psi_1(\sigma,\eta)+\nu^{3/2}\psi_2(\sigma,\eta)+\nu^{5/2}\psi_3(\sigma,\eta)+\cdots \tag{11}
$$

and this is to be substituted into equation (10) and the coefficients of explicit powers of ν set equal to zero. The function, $\nu^{1/2}\psi_1(\sigma,\eta)$, will be found to be a solution of the parabolic boundary layer problem and this scheme may be thought of as a perturbation, in ν, around the parabolic boundary layer solution. If, in equation (7), ν is set equal to zero, the order of the equation is depressed. One therefore calls this perturbation scheme singular and expects some sort of singularity in the solution (references 12 and 13). Furthermore, the expansion (11) is not a true power series in ν since the coefficients $\psi_n(\sigma,\eta)$ depend implicitly on ν through η.

Alden's scheme produces a system of equations for the $\psi_n(\sigma,\eta)$ and for each equation the variables may be separated if one sets

$$\psi_n(\sigma,\eta) = (u_\infty \sigma^2)^{\frac{3-2n}{2}} f_n(\xi) \tag{12}$$

This yields a system of ordinary differential equations for the $f_n(\xi)$. Unfortunately expression (12) implies that, for $n \geqslant 2$, $\psi_n(\sigma,\eta) \rightarrow \infty$ as $\sigma \rightarrow 0$. It might be possible however, to show that the expression (11) is an asymptotic series, uniform in σ and η , in some region excluding an arbitrarily small region bounded by a parabola enclosing the stagnation line ($\sigma = 0$) and the leading edge. This introduces the further complication of whether the boundary conditions which remain, are sufficient for the problem. The region mentioned above excludes the origin, which is the most likely point where a non-uniformity will occur in the solution.

The iteration scheme yields the following equations for the $f_n(\xi)$

$$(f_1''' + f_1 f_1'')' = 0 \tag{13}$$

and for $n = 2, 3, 4, \ldots$

$$f_n'''' + f_1 f_n''' + (2n-1) f_1' f_n'' + f_1'' f_n' + (3-2n) f_1''' f_n = F_n(\xi) \tag{14}$$

where $F_n(\xi)$ is dependent on

$\xi, f_1, f_2, \cdots$ and f_{n-1} ,

and the primes indicate differentiation with respect to ξ .

($F_2(\xi)$, $F_3(\xi)$, and $F_4(\xi)$ are exhibited in Appendix I).

The boundary conditions (5) require the following behavior for the $f_n(\xi)$, introduced in equations (11) and (12),

$$\left.\begin{aligned} f_n(0) &= 0 \\ f_n'(0) &= 0 \end{aligned}\right\} \quad n = 1, 2, 3, \ldots \tag{16}$$

$$\left.\begin{array}{ll} f_1(\xi)/2\xi & \rightarrow 1 \\ f_1'(\xi)/2 & \rightarrow 1 \\ \left.\begin{array}{ll} f_n(\xi)/\xi & \rightarrow 0 \\ f_n'(\xi) & \rightarrow 0 \end{array}\right\} n = 2, 3, 4, \cdots & \end{array}\right\} \text{as } \xi \rightarrow \infty \tag{16}$$

From equation (13)

$$f_1''' + f_1 f_1'' = C_{11}, \text{ a constant.} \tag{17}$$

Alden states that because of conditions (16)

$$f_1'''(\xi) \rightarrow 0 \text{ and } f_1(\xi) f_1''(\xi) \rightarrow 0 \text{ as } \xi \rightarrow \infty,$$

This can be proved by making the highly reasonable assumption that the velocity gradients are continuous. Because of this hypothesis, $f_1(\xi)$ and its first two derivatives are continuous on $0 \leq \xi < \infty$. Now, rewrite equation (17) as follows.

$$\frac{f_1''(\xi)}{\xi} + \frac{f_1(\xi) f_1'(\xi)}{\xi} = \frac{C_{12}}{\xi} + C_{11} + \frac{1}{\xi}\int_0^\xi [f_1'(\eta)]^2 d\eta \tag{18}$$

Using conditions (16), it is easy to show that

$$\lim_{\xi \rightarrow \infty} \frac{f_1''(\xi)}{\xi} = C_{11}$$
$$\lim_{\xi \rightarrow \infty} \frac{f_1'(\xi)}{\xi^2} = C_{11} = 0 \tag{19}$$

Thus, the equation for f_1 becomes

$$f_1''' + f_1 f_1'' = 0 \tag{20}$$

This equation with the conditions in equations (16) possesses a solution (reference 14), often referred to as the Blasius function, which is, together with all of its derivatives, continuous. Furthermore, this is the only solution with continuous derivatives (reference 14).

The Blasius function exhibits the following asymptotic behavior as $\xi \rightarrow \infty$ (reference 15).

$$f_1(\xi) \backsim 2\xi - \beta \,, \qquad \beta = 1.73$$
$$f_1'(\xi) \backsim 2 \qquad (21)$$
$$f_1^{(n)}(\xi) \backsim 0 \qquad , \qquad n = 2, 3, 4, \ldots$$

Using equations (6), (9) and (12), one may formally rewrite expression (11):

$$\psi(\sigma, \eta, \nu) = \frac{u_\infty y}{2\xi} \sum_{n=0}^{\infty} f_{n+1}(\xi) \left(\frac{\nu}{u_\infty \sigma^2}\right)^n \qquad (22)$$

If conditions (16) are satisfied and each $f_n(\xi)$ is continuous on $0 \leq \xi < \infty$, then $u_\infty y f_n(\xi)/2\xi$ is bounded on $0 \leq \xi < \infty$ and expression (22) is an asymptotic expansion.

Unfortunately, Alden's scheme does not lead to a solution of the Navier-Stokes problem since $f_4(\xi)$ cannot be made to satisfy conditions (16). In particular, it is impossible to choose $f_4(\xi)$ so that $\lim_{\xi \to \infty} \frac{f_4(\xi)}{\xi} = 0$. This will be shown in the following pages.

Let
$$C_{n1} = f_n'''(0) \qquad (23)$$
$$C_{n2} = f_n''(0)$$

One may rewrite equation (14) as

$$f_n''' + f_1 f_n'' + (2n-2) f_1' f_n' + (3-2n) f_1'' f_n = F_{n1}(\xi) \qquad (24)$$
$$F_{n1}(\xi) = C_{n1} + \int_0^\xi F_n(\eta)\, d\eta$$

Integrating again,

$$f_n'' + f_1 f_n' + (2n-3) f_1' f_n = F_{n2}(\xi) \qquad (25)$$
$$F_{n2}(\xi) = C_{n2} + \int_0^\xi \left[F_{n1}(\eta) + 2(2n-3) f_1''(\eta) f_n(\eta)\right] d\eta$$

Let

$$h_n(\xi) = \exp\left[(\xi - \beta/2)^2\right] \frac{d^{2n-4}}{d\xi^{2n-4}} \left[-(\xi - \beta/2)^2\right] \qquad (26)$$

for $n = 2, 3, 4, \ldots$

Multiply equation (25) by $h_n(\xi)$ and obtain

$$[h_n f_n' + (h_n f_1 - h_n')f_n]' + f_n[h_n'' - f_1 h_n' + 2(n-2)f_1' h_n] = h_n F_{n2} \,. \tag{27}$$

One can show that

$$\begin{aligned} h_n f_n' + (h_n f_1 - h_n')f_n &= F_{n3}(\xi), \\ F_{n3}(\xi) &= \int_0^\xi \Big[h_n(\eta)F_{n2}(\eta) + f_n(\eta)\{(f_1(\eta) - 2\eta + \beta)h_n'(\eta) \\ &\qquad + 2(n-2)(2 - f_1'(\eta))h_n(\eta)\}\Big]\, d\eta \,, \end{aligned} \tag{28}$$

and that α_n may be chosen so, that, for $\alpha_n \leq \xi < \infty$,

$$\begin{aligned} \left[\frac{e^{(\xi-\beta/2)^2}}{h_n} f_n\right]' &= \frac{e^{(\xi-\beta/2)^2}}{h_n^2}\Big[h_n f_n' + (2\xi - \beta)h_n f_n - h_n' f_n\Big] \\ &= \frac{e^{(\xi-\beta/2)^2}}{h_n^2}\Big[F_{n3}(\xi) + (2\xi - \beta - f_1)h_n f_n\Big] \,. \end{aligned} \tag{29}$$

$$\begin{aligned} f_n(\xi) &= C_{n4} h_n(\xi) e^{-(\xi-\beta/2)^2} \\ &\qquad + h_n(\xi) e^{-(\xi-\beta/2)^2} \int_{\alpha_n}^{\xi} e^{(\eta-\beta/2)^2} F_{n4}(\eta)\, d\eta \,, \\ F_{n4}(\xi) &= \frac{1}{[h_n(\xi)]^2}\Big[F_{n3}(\xi) + h_n(\xi) f_n(\xi)(2\xi - \beta - f_1(\xi))\Big] \,. \end{aligned} \tag{30}$$

The $h_n(\xi)$ and $f_n(\xi)$ are asymptotically finite, i.e., for each n there is an integer q such that $h_n(\xi)/\xi^q \to 0$ as $\xi \to \infty$.

$$
\begin{aligned}
f_n(\xi) &\backsim h_n(\xi)\, e^{-(\xi-\beta/2)^2} \int_{\alpha_n}^{\xi} e^{(\eta-\beta/2)^2} F_{n4}(\eta)\, d\eta \\
F_{n4}(\xi) &\backsim \left[h_n(\xi)\right]^{-2} F_{n3}(\xi) \\
F_{n3}(\xi) &\backsim \int_0^{\xi} h_n(\eta)\, F_{n2}(\eta)\, d\eta \;+\; \text{a constant} \qquad (31) \\
F_{n2}(\xi) &\backsim C_{n2} + \int_0^{\xi} F_{n1}(\eta)\, d\eta \;+\; \text{a constant} \\
F_{n1}(\xi) &\backsim C_{n1} + \int_0^{\xi} F_n(\eta)\, d\eta
\end{aligned}
$$

Carrying out the obvious algebra, one can show (see Appendix II) that

$$f_2(\xi) \backsim K\, e^{-(\xi-\beta/2)^2} \int_0^{\xi} e^{(\eta-\beta/2)^2} d\eta \qquad (32)$$

and

$$\lim_{\xi\to\infty} \frac{f_4(\xi)}{\xi^2 \ln \xi} = 6\beta K \qquad (33)$$

Expression (32) is identical with Alden's result and is introduced here to show that $K \neq 0$. In fact Alden computed K =1.660 by numerical integration. Thus, $f_4(\xi)$ cannot be made to satisfy condition (16) and consequently the proposed expressions (11) and (12) cannot be made, even formally, to satisfy equations (10) and (5). It may be remarked that the Alden approach with a slip boundary condition is subject to the same difficulty.

2.2 The Flat Plate Boundary Layer Problem With Slip

In an attempt to incorporate rarefaction effects into boundary layer theory, one is led (reference 9)to consider a slip boundary condition, $u = \zeta \frac{\partial u}{\partial y}$, where ζ will be called the slip parameter. ζ is proportional to the mean free path length.

The flat plate boundary layer problem with slip (i.e., with a slip boundary condition) has not yet been solved. With the idea of gaining some knowledge of the slip effect in general, the corresponding linearized problem for incompressible flow, which has been solved explicitly (reference 7), will now be examined. First, the problem will be formulated and the solution displayed. Secondly, a series solution of restricted validity will be obtained by a perturbation scheme in ζ. Lastly, these solutions will be compared and analyzed.

The linearized, incompressible, steady-state boundary layer problem with slip, for the semi-infinite flat plate at zero angle of attack may be stated

$$\rho_\infty u_\infty \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \tag{34}$$

$$u \rightarrow u_\infty \quad \text{as} \quad y \rightarrow \infty, \quad x > 0 \tag{35}$$

$$u = \zeta \frac{\partial u}{\partial y} \quad \text{at} \quad y = 0, \quad x > 0 \tag{36}$$

This is reminiscent of the Rayleigh impulsive plate problem, and if condition (35) is replaced by

$$\left.\begin{array}{l} u = u_\infty \quad \text{at} \quad x = 0, \quad y > 0 \\ u \text{ bounded in the entire plane} \end{array}\right\} \tag{37}$$

one may solve this new problem with the aid of Laplace transforms (reference 7). Fortunately, the solution to this second problem satisfies conditions (35) so the solution to the original problem is at hand. This solution is:

$$u = u_\infty \left[\operatorname{erf}\left(\sqrt{\frac{u_\infty y^2}{4\nu_\infty x}}\right) + \exp\left(\frac{\nu_\infty x}{u_\infty \zeta^2} + \frac{y}{\zeta}\right) \operatorname{erfc}\left(\sqrt{\frac{\nu_\infty x}{u_\infty \zeta^2}} + \sqrt{\frac{u_\infty y^2}{4\nu_\infty x}}\right)\right] \tag{38}$$

$$\frac{\partial u}{\partial x} = u_\infty \left[\frac{\nu_\infty}{u_\infty \zeta} \exp\left(\frac{\nu_\infty x}{u_\infty \zeta^2} + \frac{y}{\zeta}\right) \operatorname{erfc}\left(\sqrt{\frac{\nu_\infty x}{u_\infty \zeta^2}} + \sqrt{\frac{u_\infty y^2}{4\nu_\infty x}}\right) \right.$$
$$\left. - \frac{1}{\sqrt{\pi}} \sqrt{\frac{\nu_\infty}{u_\infty \zeta^2 x}} \exp\left(\frac{-u_\infty y^2}{4\nu_\infty x}\right)\right]. \tag{39}$$

$$\frac{\partial u}{\partial y} = u_\infty \left[\frac{1}{\zeta} \exp\left(\frac{\nu_\infty x}{u_\infty \zeta^2} + \frac{y}{\zeta} \right) \operatorname{erfc}\left(\sqrt{\frac{\nu_\infty x}{u_\infty \zeta^2}} + \sqrt{\frac{u_\infty y^2}{4 \nu_\infty x}} \right) \right] \tag{40}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_\infty}{\nu_\infty} \frac{\partial u}{\partial x} \tag{41}$$

If one defines a local skin friction coefficient,

$$C_f = \frac{\mu}{\frac{1}{2} \rho_\infty u_\infty^2} \left(\frac{\partial u}{\partial y} \right)_{y=0} \tag{42}$$

one obtains, for this problem,

$$C_f = \frac{2 \nu_\infty}{u_\infty \zeta} \exp\left(\frac{\nu_\infty x}{u_\infty \zeta^2} \right) \operatorname{erfc}\left(\sqrt{\frac{\nu_\infty x}{u_\infty \zeta^2}} \right) \tag{43}$$

The drag coefficient for that portion of the plate extending from the origin to a distance L downstream will be defined (including both upper and lower plate surfaces) as

$$C_D = \frac{2}{L} \int_0^L C_f \, dx \tag{44}$$

and for this problem

$$C_D = 4 \sqrt{\frac{\nu_\infty}{u_\infty L}} \left\{ \sqrt{\frac{u_\infty \zeta^2}{\nu_\infty L}} \left[\exp\left(\frac{\nu_\infty L}{u_\infty \zeta^2} \right) \operatorname{erfc}\left(\sqrt{\frac{\nu_\infty L}{u_\infty \zeta^2}} \right) - 1 \right] + \frac{2}{\sqrt{\pi}} \right\} \tag{45}$$

Equations (38), (43), (45) constitute the solution of the originally posed problem.

An attempt to solve this same problem by a formal iteration in ζ, commences with the formal series

$$u(x, y, \zeta) = u_0(x, y) + \zeta u_1(x, y) + \zeta^2 u_2(x, y) + \cdots \tag{46}$$

This yields the iteration scheme

$$\left.\begin{aligned} &\rho_\infty u_\infty \frac{\partial u_0}{\partial x} = \mu \frac{\partial^2 u_0}{\partial y^2} \\ &u_0 \to u_\infty \quad \text{as} \quad y \to \infty, \; x > 0 \\ &u_0 = 0 \quad \text{at} \quad y = 0, \; x \geq 0 \end{aligned}\right\} \tag{47}$$

and for $n = 1, 2, 3, \ldots$

$$\left.\begin{aligned} &\rho_\infty u_\infty \frac{\partial u_n}{\partial x} = \mu \frac{\partial^2 u_n}{\partial y^2} \\ &u_n \to 0 \quad \text{as} \quad y \to \infty, \; x > 0 \\ &u_n = \frac{\partial u_{n-1}}{\partial y} \quad \text{at } y = 0, \; x > 0 \end{aligned}\right\} \tag{48}$$

The technique employed in solving the problem delineated by equation (34), under the conditions (35) and (36) may be employed here to solve equations (47). The no-slip solution, $u_0(x, y)$ is

$$u_0(x,y) = u_\infty \operatorname{erf}\left(\sqrt{\frac{u_\infty y^2}{4\nu_\infty x}}\right) \tag{49}$$

It is easy to see that

$$u_n(x,y) = \frac{\partial u_{n-1}(x,y)}{\partial y} \qquad \text{for } n = 1, 2, 3, \ldots \tag{50}$$

solves equations (48). Thus, formally,

$$u(x,y,\zeta) = \sum_{n=0}^{\infty} \zeta^n \frac{\partial^n}{\partial y^n}\left(u_\infty \operatorname{erf}\left(\sqrt{\frac{u_\infty y^2}{4\nu_\infty x}}\right)\right) \tag{51}$$

From the definition (42) and equation (51) one obtains, formally

$$C_f = \frac{\mu}{\frac{1}{2}\rho_\infty u_\infty^2} \sum_{n=0}^{\infty} \zeta^n \left[\frac{\partial^{n+1}}{\partial y^{n+1}}\left(u_\infty \operatorname{erf}\left(\sqrt{\frac{u_\infty y^2}{4\nu_\infty x}}\right)\right)\right]_{y=0}$$

$$= \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu_\infty}{u_\infty x}} \sum_{n=0}^{\infty} \zeta^n \left[\frac{\partial^n}{\partial y^n} \exp\left(\frac{-u_\infty y^2}{4\nu_\infty x}\right)\right]_{y=0} \tag{52}$$

$$= \frac{2}{\pi}\sqrt{\frac{\nu_\infty}{u_\infty x}}\sum_{n=0}^{\infty}(-1)^n\,\Gamma\left(\frac{2n+1}{2}\right)\left(\frac{u_\infty \zeta^2}{\nu_\infty x}\right)^n$$

Integration by parts and the use of induction on expression (43) yields

$$C_f \sim \frac{2}{\pi}\sqrt{\frac{\nu_\infty}{u_\infty x}}\sum_{n=0}^{\infty}(-1)^n\,\Gamma\left(\frac{2n+1}{2}\right)\left(\frac{u_\infty \zeta^2}{\nu_\infty x}\right)^n \tag{53}$$

Thus, one sees that the formal iteration scheme led to an asymptotic (around $\zeta = 0$) expansion for C_f (which is uniform in x for x bounded away from zero). Expression (52) is useless at $x = 0$ and cannot help in the determination of C_D; it can, however, in a series of unjustified steps, be summed exponentially, to yield expression (43) (see Appendix III).

From expression (43), one learns that

$$\left.\begin{aligned} &\lim_{x \to 0} C_f = \frac{2\nu_\infty}{u_\infty \zeta} && \text{for fixed } \zeta > 0 \\ &\lim_{\zeta \to 0} C_f = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu_\infty}{u_\infty x}} && \text{for fixed } x > 0 \end{aligned}\right\} \tag{54}$$

and the non-uniformity of C_f in ζ and x at $\zeta = x = 0$ is apparent. From expression (53),

$$C_f \sim \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu_\infty}{u_\infty x}}\left[1 - \frac{1}{2}\frac{u_\infty \zeta^2}{\nu_\infty x} + \frac{3}{4}\left(\frac{u_\infty \zeta^2}{\nu_\infty x}\right)^2 + \cdots\right] \tag{55}$$

Integration by parts and the use of induction on expression (45) yields

$$C_D \sim \frac{8}{\sqrt{\pi}}\sqrt{\frac{\nu_\infty}{u_\infty L}}\left[1 - \frac{\sqrt{\pi}}{2}\sqrt{\frac{u_\infty \zeta^2}{\nu_\infty L}} + \frac{1}{2}\frac{u_\infty \zeta^2}{\nu_\infty L} - \frac{1}{4}\left(\frac{u_\infty \zeta^2}{\nu_\infty L}\right)^2 + \cdots\right] \tag{56}$$

From equations (55) and (56) it is evident that a slip correction to the local skin friction of order ζ^2 does not imply that the slip correction to the drag is also of order ζ^2, since the non-uniformity is sufficiently strong so that the actual correction is of order ζ.

3.0 CONCLUSIONS

1. The scheme, which Alden used to correct the Blasius solution, cannot lead to a solution, even formally, of the incompressible Navier-Stokes semi-infinite flat plate problem.
2. For the incompressible linearized flat plate problem the inclusion of a slip boundary condition introduces a correction to the local skin friction of order ζ^2 but, due to a non-uniformity in the solution, corrects the drag coefficient to order ζ .

APPENDIX I

$$F_2(\xi) = 4\xi f_1''' - \xi^2 f_1'''' - \xi^2 f_1 f_1''' - 2\xi f_1 f_1''' + \xi^2 f_1' f_1''$$

$$\begin{aligned}
F_3(\xi) = {} & -3 f_2' f_2'' + f_2 f_2''' - 10 f_1' f_2 - \xi^2 f_1' f_1'' - 2 f_1 f_2' - \xi^2 f_1 f_2''' \\
& -2\xi f_1'' f_2 + \xi^2 f_2' f_1'' + \xi^2 f_2 f_1''' + 2\xi f_2'' f_1 - 12 f_2'' \\
& + 4\xi f_2''' - \xi^2 f_2''''
\end{aligned}$$

$$\begin{aligned}
F_4(\xi) = {} & -3 f_2' f_3'' + f_2 f_3''' - 2 f_2'' f_3' - f_3' f_2'' + 3 f_3 f_2''' + 2 f_3'' f_2' \\
& -60 f_1' f_3 - 3\xi^2 f_1' f_3'' - 12 f_1 f_3' - \xi^2 f_1 f_3''' - 6\xi f_1'' f_3 \\
& -6 f_2' f_2 - \xi^2 f_2' f_2'' + 2 f_2 f_2' + \xi^2 f_2 f_2''' - 4 f_2 f_2' - 2\xi f_2'' f_2 \\
& + \xi^2 f_3' f_1'' + 3\xi^2 f_3 f_1''' - 24 f_3 f_1' + 2\xi f_3'' f_1 - 6\xi^2 f_1' f_2 \\
& -2\xi^2 f_1 f_2' + 4\xi f_2 f_1 - 40 f_3'' + 4\xi f_3''' - \xi^2 f_3'''' - 56 f_2 \\
& + 8\xi f_2' - 4\xi^2 f_2''
\end{aligned}$$

APPENDIX II

From Appendix I and equation (21) $F_2 \backsim 0$. Introducing the integration constants, K_{mn} , one obtains

$$
\begin{aligned}
F_2 &\backsim 0 \\
F_{21} &\backsim C_{21} + K_{21} \\
F_{22} &\backsim (C_{21} + K_{21})(\xi - \beta/2) + C_{22} + K_{22} \\
h_2 &= 1 \\
F_{23} &\backsim (C_{21} + K_{21}) \frac{(\xi - \beta/2)^2}{2} + (C_{22} + K_{22})(\xi - \beta/2) + K_{23} \\
F_{24} &\backsim F_{23} \\
f_2 &\backsim \frac{C_{21} + K_{21}}{2}\left[\frac{(\xi - \beta/2)}{2} + O(\xi^{-1})\right] + \frac{C_{22} + K_{22}}{2} \\
&\quad + \frac{K_{23}}{2}\left[(\xi - \beta/2)^{-1} + \frac{(\xi - \beta/2)^{-3}}{2}\right] + O(\xi^{-5})
\end{aligned}
\tag{II-1}
$$

Because of conditions (16), one must put

$$C_{21} = - K_{21} \tag{II-2}$$

Using equations (14), (24), (25) and (28), one obtains

$$
\begin{aligned}
f_2 &= \frac{C_{22} + K_{22}}{2} + K_{23}\left[\frac{(\xi - \beta/2)^{-1}}{2} + \frac{(\xi - \beta/2)^{-3}}{4}\right] + O(\xi^{-5}) \\
f_2' &= -K_{23}\left[\frac{(\xi - \beta/2)^{-2}}{2} + \frac{3(\xi - \beta/2)^{-4}}{4}\right] + O(\xi^{-6}) \\
f_2'' &= K_{23}\left[(\xi - \beta/2)^{-3} + 3(\xi - \beta/2)^{-5}\right] + O(\xi^{-7}) \\
f_2''' &= -K_{23}\left[3(\xi - \beta/2)^{-4} + 15(\xi - \beta/2)^{-6}\right] + O(\xi^{-8}) \\
f_2'''' &= K_{23}\left[12(\xi - \beta/2)^{-5} + 90(\xi - \beta/2)^{-7}\right] + O(\xi^{-9})
\end{aligned}
\tag{II-3}
$$

Appendix I and equations (21) and (II-3) yield

$$F_3 = -10(C_{22} + K_{22}) + 6\beta K_{23} (\xi - \beta/2)^{-2} + O(\xi^{-3}) \qquad \text{(II-4)}$$

Proceeding in the same way as with F_2 ,

$$F_{31} = -10(C_{22} + K_{22})(\xi - \beta/2) + C_{31} + K_{31}$$
$$- 6\beta K_{23} (\xi - \beta/2)^{-1} + O(\xi^{-2})$$

$$F_{32} = -5(C_{22} + K_{22})(\xi - \beta/2)^2 + (C_{31} + K_{31})(\xi - \beta/2)$$
$$- 6\beta K_{23} \ln(\xi - \beta/2) + O(\xi^{-1})$$

$$h_3 = 4(\xi - \beta/2)^2 - 2 \qquad \text{(II-5)}$$

$$F_{33} = -4(C_{22} + K_{22})(\xi - \beta/2)^5 \left[1 + O(\xi^{-2})\right]$$
$$+ (C_{31} + K_{31})(\xi - \beta/2)^4 \left[1 + O(\xi^{-2})\right]$$
$$- 8\beta K_{23} (\xi - \beta/2)^3 \ln(\xi - \beta/2) \left[1 + O(\xi^{-2})\right] + O(\xi^3)$$

$$\left[h_3(\xi)\right]^{-2} = \frac{1}{16}(\xi - \beta/2)^{-4} + O(\xi^{-6})$$

$$F_{34} = -\frac{1}{4}(C_{22}+K_{22})(\xi-\beta/2)\left[1+O(\xi^{-2})\right] + \frac{(C_{31}+K_{31})}{16}\left[1+O(\xi^{-2})\right]$$
$$-\frac{1}{2}\beta K_{23}(\xi-\beta/2)^{-1}\ln(\xi-\beta/2)\left[1+O(\xi^{-2})\right]$$
$$+O(\xi^{-1})$$

$$f_3 = -\frac{1}{2}(C_{22}+K_{22})(\xi-\beta/2)^2\left[1+O(\xi^{-2})\right]$$
$$+\frac{(C_{31}+K_{31})}{8}(\xi-\beta/2)\left[1+O(\xi^{-2})\right] \qquad \text{(II-6)}$$
$$-\beta K_{23}\ln(\xi-\beta/2)\left[1+O(\xi^{-2})\right]$$
$$+O(1)$$

Because of conditions (16),

$$C_{22} = -K_{22}$$
$$C_{31} = -K_{31} \qquad \text{(II-7)}$$

$$f_3 = -\beta K_{23}\ln(\xi-\beta/2) + O(1) + \ln(\xi-\beta/2)\,O(\xi^{-2})$$
$$f_3' = -\beta K_{23}(\xi-\beta/2)^{-1} + O(\xi^{-2}) + \ln(\xi-\beta/2)\,O(\xi^{-3})$$
$$f_3'' = \beta K_{23}(\xi-\beta/2)^{-2} + O(\xi^{-3}) + \ln(\xi-\beta/2)\,O(\xi^{-4}) \qquad \text{(II-8)}$$
$$f_3''' = -2\beta K_{23}(\xi-\beta/2)^{-3} + O(\xi^{-4}) + \ln(\xi-\beta/2)\,O(\xi^{-5})$$
$$f_3'''' = 6\beta K_{23}(\xi-\beta/2)^{-4} + O(\xi^{-5}) + \ln(\xi-\beta/2)\,O(\xi^{-6})$$

$$F_4 = 168\beta K_{23} \ln(\xi - \beta/2) + O(1) + \ln(\xi - \beta/2)\, O(\xi^{-2})$$

$$F_{41} = 168\beta K_{23} (\xi - \beta/2) \ln(\xi - \beta/2) + O(\xi) + \ln(\xi - \beta/2)\, O(1)$$

$$F_{42} = 84\beta K_{23} (\xi - \beta/2)^2 \ln(\xi - \beta/2) + [\ln(\xi - \beta/2)]^2 O(1)$$
$$+ O(\xi^2) + \ln(\xi - \beta/2)\, O(\xi)$$

$$h_4 = 16(\xi - \beta/2)^4 - 48(\xi - \beta/2)^2 + 12$$

$$F_{43} = 192\beta K_{23} (\xi - \beta/2)^7 \ln(\xi - \beta/2)$$
$$+ [\ln(\xi - \beta/2)]^2 O(\xi^5) + O(\xi^7)$$
$$+ \ln(\xi - \beta/2)\, O(\xi^6)$$

$$F_{44} = \frac{3}{4}\beta K_{23} (\xi - \beta/2)^{-1} \ln(\xi - \beta/2) \tag{II-9}$$
$$+ [\ln(\xi - \beta/2)]^2 O(\xi^{-3}) + O(\xi^{-1})$$
$$+ \ln(\xi - \beta/2)\, O(\xi^{-2})$$

$$f_4 = 6\beta K_{23} (\xi - \beta/2)^2 \ln(\xi - \beta/2)$$
$$+ [\ln(\xi - \beta/2)]^2 O(1) + O(\xi^2)$$
$$+ \ln(\xi - \beta/2)\, O(\xi)$$

$$[h_4(\xi)]^{-2} = \frac{1}{256} (\xi - \beta/2)^{-8} + O(\xi^{-10})$$

APPENDIX III

Proceeding from equation (52) in a purely formal manner:

$$C_f = \frac{2}{\pi}\sqrt{\frac{\nu_\infty}{u_\infty x}}\sum_{n=0}^{\infty}\Gamma\left(\frac{2n+1}{2}\right)\left(\frac{-u_\infty \zeta^2}{\nu_\infty x}\right)^n\left[\frac{1}{\Gamma(n+1)}\int_0^\infty e^{-t}t^n\,dt\right]$$

$$= \frac{2}{\pi}\sqrt{\frac{\nu_\infty}{u_\infty x}}\int_0^\infty e^{-t}\left[\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}\left(\frac{-u_\infty \zeta^2 t}{\nu_\infty x}\right)^n\right]dt \tag{III-1}$$

$$= \frac{2}{\pi}\frac{\nu_\infty}{u_\infty \zeta}\sqrt{\frac{u_\infty \zeta^2}{\nu_\infty x}}\int_0^\infty e^{-t}\left[1+\frac{u_\infty \zeta^2}{\nu_\infty x}t\right]^{-\frac{1}{2}}dt$$

The expression

$$I(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-t}\left[1+\frac{t}{\lambda^2}\right]^{-\frac{1}{2}}dt \tag{III-2}$$

with

$$\lambda = \sqrt{\frac{\nu_\infty x}{u_\infty \zeta^2}} \tag{III-3}$$

satisfies

$$\frac{dI}{d\lambda} - 2\lambda I + 2 = 0 \tag{III-4}$$

$$I(0) = \sqrt{\pi}$$

Therefore

$$I(\lambda) = \sqrt{\pi}\,\exp(\lambda^2)\,\mathrm{erfc}(\lambda) \tag{III-5}$$

and finally,

$$C_f = \frac{2\nu_\infty}{u_\infty \zeta} \exp\left(\frac{\nu_\infty x}{u_\infty \zeta^2}\right) \operatorname{erfc}\left(\sqrt{\frac{\nu_\infty x}{u_\infty \zeta^2}}\right) \tag{III-6}$$

This method of "exponentially summing" a divergent series is discussed by Borel in reference 16.

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